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Electrical networks on *n*-simplex fractals

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Abstract
The decimation map $\mathcal{D}$ for a network of admittances on an n-simplex lattice fractal is studied. The asymptotic behaviour of $\mathcal{D}$ for large-size fractals is examined. It is found that in the vicinity of the isotropic point the eigenspaces of the linearized map are always three for $n \geqslant 4$; they are given a characterization in terms of graph theory. A new anisotropy exponent, related to the third eigenspace, is found, with a value crossing over from $\ln[(n+2)/3]/\ln 2$ to $\ln[(n+2)^3/n(n+1)^2]/\ln 2$.

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1. Introduction

The 'n-simplex lattice', built through the iteration of a complete graph K_n, is the generalization of the usual two-dimensional Sierpiński gasket fractal to a d-dimensional space, with $n = d+1$. It was first introduced 30 years ago by Dhar [1] (in a slightly different, 'truncated' version) as one of the first examples of lattices with nonintegral dimensionality.

The problem of *electrical networks* on n-simplex lattices was initially considered under the simple form of *isotropic resistor networks* [2], modelling the conducting backbone of a percolating system at criticality. The aim was to study the anomalous scaling law for the total conductance σ (or complex admittance, in the case of a general passive element) as a function of the size: it was found that $\sigma \sim L^{-\bar{\zeta}}$, with $\bar{\zeta} = \ln[(n+2)/n]/\ln 2$. It was soon realized [3] that on these fractals the macroscopic conductance is isotropic even starting from microscopic anisotropy. Hence, *anisotropic* networks were considered, in order to find the scaling exponent $\bar{\lambda}$ with which the conductance anisotropy (defined as the difference from 1 of the ratio of two conductances σ_α and σ_β measured along different directions) vanishes: $(\sigma_\alpha - \sigma_\beta)/\sigma_\beta \sim L^{-\bar{\lambda}}$.

The problem consists in finding the decimation transformation $\mathcal{D}$ connecting the conductances of generation $g+1$ of a fractal to those of generation g, then linearizing $\mathcal{D}$ near the isotropic point. The largest eigenvalue of the linearized map, $e_1 = n/(n+2)$, is then related to the isotropic exponent by $\bar{\zeta} = -\ln e_1/\ln 2$; the second eigenvalue e_2 is related to the anisotropy exponent by $\bar{\lambda} = \ln(e_1/e_2)/\ln 2$. Since no simple method is known for finding $\mathcal{D}$ for $n > 3$, the analytical challenges in the anisotropic case are much greater. The greatest effort in this direction at that time was that by Vannimenus and Knežević [3], who managed to find $\bar{\lambda}$ for $n = 4, n = 5$, and suggested the general form $\bar{\lambda} = \ln[(n+2)/(n+1)]/\ln 2$ (corresponding to $e_2 = n(n+1)/(n+2)^2$). More recently, in the light of a renewed interest in the restoration of isotropy in several models defined on fractals [4], Jafarizadeh [5] succeeded in proving the formula suggested in [3] for n-simplex lattices for any n.

The general problem of the decimation of a network of conductances on the n-simplex lattice had never been completely solved up to now. In a previous paper [6], we found the exact map for the decimation of a network of impedances on a three-dimensional Sierpiński gasket (4-simplex lattice), with a method based on the direct manipulation of the Laplacian matrix of the circuit. In this paper, we generalize that method and show a detailed procedure to find the decimation map $\mathcal{D}$ for arbitrary n. Instead of conductances, more general complex admittances will be used. For the first time, a systematic analysis of the asymptotic expansion of $\mathcal{D}$ near the isotropic fixed point will be given.

The main results are the following. There are always exactly *three* eigenvalues regardless of n, that is, of the dimension of the fractal (being two just for the two-dimensional system). The third eigenvalue is $e_3 = 3n/(n+2)^2$ and is related to a secondary anisotropy exponent equal to $\ln[(n+2)/3]/\ln 2$; however, this exponent crosses over to $\ln[(n+2)^3/n(n+1)^2]/\ln 2$ for large systems. The eigenspaces corresponding to e_2 and e_3 will be studied for the first time. We will show that the eigenvectors corresponding to e_2 are those that allow a direct simplified (mesh-to-star) treatment of the problem. It will also be found that the eigenspace corresponding to e_3 is related to the space of cycles of even length on the complete graph K_n, and has the highest dimensionality for n large, scaling as n^2.

The plan of the paper is as follows. In section 2, we introduce the model and the notation we will use throughout the paper. In section 3 we describe our general decimation procedure, which we apply first to the case of the isotropic fractal and, second, to that of an isotropic fractal with a small perturbation, which provides us with the linear expansion of the decimation map $\mathcal{D}$ in the vicinity of the isotropic point. Section 4 examines the eigenvalues and eigenspaces of the linearized map. Section 5 contains our conclusions.

2. Basic cell and notations. Construction of a network of admittances. Statement of the problem

The n-simplex lattice is built by the iteration of a basic cell (figure 1). The basic cell is the complete graph K_n (n-simplex), with n vertices and $n(n-1)/2$ links, one between each pair of vertices. We follow the convention to arrange the vertices of the graph as those of an n-polygon and label them with the numbers from 1 to n counterclockwise. We recall that two links are called *adjacent* when they share a vertex, and *non-adjacent* when they do not. In a K_n graph (see figure 2 for graph K_5) each link has exactly $2(n-2)$ adjacent links and $1/2(n-2)(n-3)$ non-adjacent links (corresponding to a K_{n-2} complete subgraph).

2.1. *Construction of a network of admittances. Kirchhoff's laws for the basic cell*

We consider first a *general* electrical network, which can be represented by a graph, i.e., a set of nodes connected by links, with an electrical pole on each node i and an electrical element on each link. We will consider here only linear passive elements, that is resistances, inductances and capacitances. We call $z_{(i,j)}$ the impedance of the element (if any) connecting nodes i and

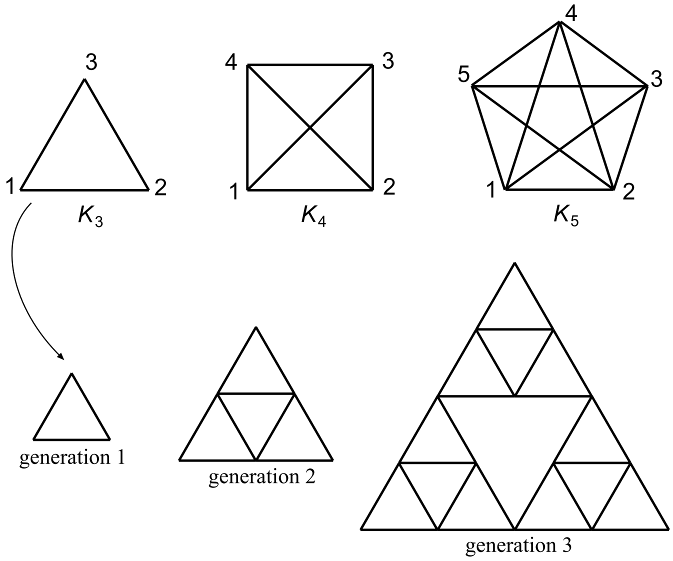

Figure 1. Top: three K_n complete graphs. The n-simplex lattice is a fractal built using a K_n as the basic cell. Bottom: the construction of the two-dimensional Sierpiński gasket, or 3-simplex lattice, starting from the basic cell.

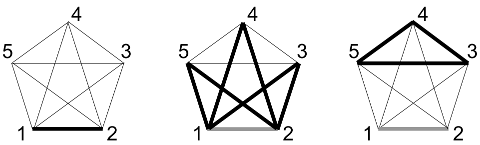

Figure 2. We consider the K_n graph and focus on one of its links (for example link (1, 2) in graph K_5 on the left). The number of adjacent links is $2(n-2)$ (middle): $n-2$ incident in vertex 1 and $n-2$ incident in vertex 2. The number of non-adjacent links is $1/2(n-2)(n-3)$ (right), that is the number of links of the K_{n-2} subgraph obtained by removing vertices 1 and 2 from K_n.

j, and $\sigma_{(i,j)} = z^{-1}_{(i,j)}$ its admittance. V_i is the potential at node i and I_i is the external current incoming at node i. The key formula we will use in the following is Kirchhoff's node law:

$$I_i = \sum_{k\neq i} \sigma_{(i,k)}(V_k - V_i) = \sum_{k\neq i} \sigma_{(i,k)} V_k - \left(\sum_{k\neq i} \sigma_{(i,k)}\right) V_i,$$

which can be expressed in a matrix form:

$$\vec{I} = \mathbf{L}\vec{V}, \tag{1}$$

where $\vec{V} = (V_1, V_2, \ldots, V_n)^{\mathrm{T}}$, $\vec{I} = (I_1, I_2, \ldots, I_n)^{\mathrm{T}}$, and $\mathbf{L}$ is the Laplacian matrix of the circuit (see e.g. [7, 8] for a recent application): $L_{ii} = \sum_{j\neq i} \sigma_{(i,j)}$ for diagonal entries, and $L_{ij} = -\sigma_{(i,j)}$ for nondiagonal entries. Since Kirchhoff's law for currents states that $\sum_i I_i = 0$, the entries of $\mathbf{L}$ are linearly dependent, and $\det(\mathbf{L}) = 0$.

We now consider the K_n graph, the basic cell (or 1st-generation fractal) of the n-simplex lattice (figure 3). The admittance between pole i and pole j is $\sigma^{(1)}_{(i,j)}$. The relation between external currents and potentials is $\vec{I} = \mathbf{L}^{(1)}\vec{V}$, where $\mathbf{L}^{(1)}$ is the Laplacian matrix of the 1st-generation fractal:

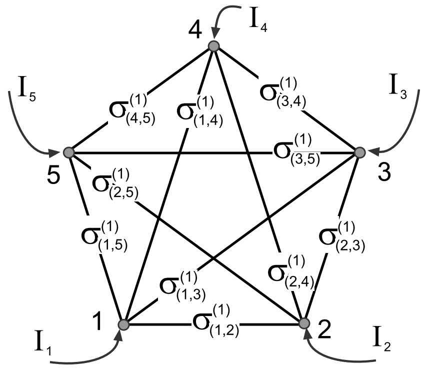

Figure 3. The K_5 graph and the labelling used. External potentials are not shown.

$$\begin{cases} L_{ii}^{(1)} = \sum_{j\neq i} \sigma_{(i,j)}^{(1)} & \text{diagonalentries} \\ L_{ij}^{(1)} = -\sigma_{(i,j)}^{(1)} & i \neq j. \end{cases} \tag{2}$$

If we start from the Laplacian matrix, the admittances of the network are given by its off-diagonal terms: $\sigma_{(i,j)}^{(1)} = -L_{ij}^{(1)}$. This relation will play an important role in our following calculations.

In several expressions in the following we will sum over *links* rather than over *vertices*: thus, we will find it useful to label links with numbers from 1 to $n(n-1)/2$, rather than with pairs. In that case we will use greek letters: $(i, j) \equiv \alpha$, with $\alpha = 1, 2, \ldots, n(n-1)/2$; hence, instead of $\sigma_{(i,j)}^{(1)}$, the form $\sigma_{\alpha}^{(1)}$ will also be used.

2.2. Construction of a network of admittances: higher generations and the decimation map

Given the 1st-generation fractal, the 2nd-generation fractal is built as follows. n samples of generation 1 are arranged counterclockwise, and labelled with the same numbers as the poles of the basic graph (see figure 4, left, for graph K_5). Then we connect (short-circuit) pole i of basic graph j with pole j of basic graph i, with $j \neq i$; such connected poles are termed the 'internal poles' of the resulting graph; the related potentials are called $v_{(i,j)}$ (note that there is a one-to-one correspondence between the internal potentials of the second-order fractal and the links of the basic graph). The poles that are left free from this procedure, i.e. every pole i of graph i, are the external poles of the 2nd-generation fractal, with their incoming currents I_i and external potentials V_i.

This circuit (figure 4) is equivalent to that built on a 1st-generation fractal, whose admittances we will call $\sigma_{(i,j)}^{(2)}$ (the equivalence comes from the fact that both circuits have $n(n-1)/2$ degrees of freedom). In general, $\sigma_{(i,j)}^{(g)}$ will be the admittances of the 1st-generation circuit equivalent to that built on a gth-generation fractal with $\sigma_{(i,j)}^{(1)}$ on the basic cell. We want to find the decimation map $\mathcal{D}$ connecting the $\{\sigma_{(i,j)}^{(1)}\}$ to the $\{\sigma_{(i,j)}^{(2)}\}$:

$$\vec{\sigma}^{(2)} = \mathcal{D}(\vec{\sigma}^{(1)}), \tag{3}$$

so that for a gth-generation fractal

$$\vec{\sigma}^{(g)} = \mathcal{D}^g(\vec{\sigma}^{(1)}).$$

In two dimensions the decimation is carried out by means of the star-delta (or star-mesh) transformations from electrostatics [9] (figure 5). Given a triangle-shaped circuit, it is always

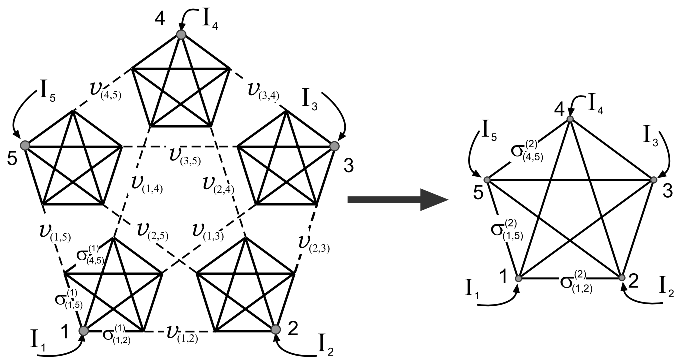

Figure 4. Left: construction of the 2nd-generation fractal starting from a basic cell with admittances $\sigma^{(1)}_{(i,j)}$. Dashed lines denote short circuits between poles, coinciding with internal poles. The internal potentials are called $v_{(i,j)}$; they are in a one-to-one correspondence with the links of the basic cell. The external potentials are not shown. Right: the circuit built on the 2nd-generation fractal is equivalent to that built of a 1st-generation fractal with new admittances $\sigma^{(2)}_{(i,j)}$.

possible to find an equivalent star-shaped circuit, with an additional central pole, and new admittances that depend on the old ones via easy algebraic relations. The vice versa (star-to-mesh) is also always possible. For a general complete graph we can define a generalized version of the star-mesh transformation: given a complete K_n graph (a generalized mesh), find the star graph with an additional hub pole that is equivalent to the mesh (mesh-to-star); or vice versa (star-to-mesh).

It turns out, however [10], that for $n > 3$ only the star-to-mesh transformation is always possible, while the mesh-to-star transformation is possible only under very restrictive conditions. These are the so-called *Wheatstone conditions*: in every four-link subgraph of the mesh forming a quadrilateral, the products of opposite admittances must be equal. Since these conditions are not satisfied in general, a different approach must be sought. Our approach, based on the direct manipulation of the Laplacian matrix of the circuit, has been used in [6] for $n = 4$. Here we show a general method valid for any n.

3. Decimation: general method, fixed points and asymptotic expansion

3.1. General procedure

The Laplacian matrix of the 2nd-generation fractal is

$$\mathbf{L^{(2)}} = \begin{array}{c} \overbrace{\qquad\qquad}^{i=1,\ldots,n} \; \overbrace{\qquad\qquad\qquad}^{\alpha=1,\ldots,n\,(n-1)/2} \\ \left(\begin{array}{c|c} \mathbf{D} & \mathbf{\Sigma^T} \\ \hline \mathbf{\Sigma} & \mathbf{M} \end{array}\right) \end{array}. \tag{4}$$

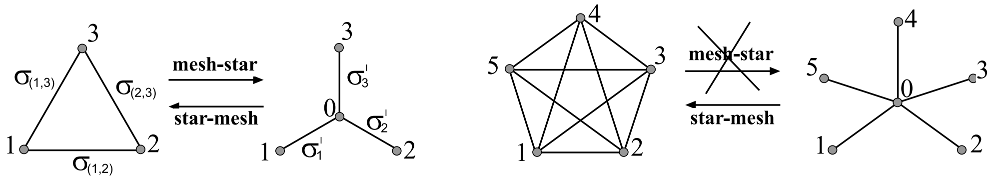

Figure 5. A circuit built on a 3-simplex (a triangle) is always equivalent to another circuit built on a star (left): this is the star-triangle, or star-mesh, transformation. For a generic n-simplex lattice (right) a generalized star-to-mesh transformation is still valid, while the mesh-to-star transformation holds only under Wheatstone's conditions.

We use an indexing such that the first n indices of a row (or column) refer to external points (and are labelled as the *nodes* of the basic cell), the last $n(n-1)/2$ refer to internal points (and are labelled as the *links* of the basic cell).

$\mathbf{D}$ is a diagonal $n \times n$ matrix with entries $D_{ii} = \sum_{j \neq i} \sigma^{(1)}_{(i,j)}$ (since an external pole has no links with other external poles).

Matrix $\mathbf{\Sigma}$, which has internal pole labels (links of the basic cell) as row indices and external pole labels (nodes of the basic cell) as column indices, has only two entries different from zero:

$$\begin{cases} \Sigma_{\alpha k} = -\sigma^{(1)}_{\alpha} & \text{for } \alpha \text{ incident in } k \text{ in the basic cell} \\ \Sigma_{\alpha k} = 0 & \text{otherwise.} \end{cases} \tag{5}$$

$\mathbf{M}$ is a $[n(n-1)/2] \times [n(n-1)/2]$ symmetric matrix. Its entries correspond to internal poles (hence, to links of the basic cell). $\mathbf{M}$ has entries

$$\begin{cases} M_{\alpha\alpha} = 2\sigma^{(1)}_{\alpha} + \displaystyle\sum_{\beta \text{ adjacent to } \alpha} \sigma^{(1)}_{\beta} & \text{for diagonal entries} \\ M_{\alpha\beta} = -\sigma^{(1)}_{\beta} & \text{if links } \alpha \text{ and } \beta \text{ of the basic cell are adjacent} \\ M_{\alpha\beta} = 0 & \text{if links } \alpha \text{ and } \beta \text{ of the basic cell are not adjacent.} \end{cases} \tag{6}$$

Kirchhoff's equations for the graph are:

$$\begin{pmatrix} \mathbf{D} & \mathbf{\Sigma^T} \\ \mathbf{\Sigma} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \vec{V} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \mathbf{D}\vec{V} + \mathbf{\Sigma^T}\vec{v} \\ \mathbf{\Sigma}\vec{V} + \mathbf{M}\vec{v} \end{pmatrix} = \begin{pmatrix} \vec{I} \\ 0 \end{pmatrix}.$$

Hence, we have two sets of equations. From the second set (regarding the internal poles), we get $\mathbf{M}\vec{v} = -\mathbf{\Sigma}\vec{V}$, and we can find the internal potentials as functions of $\sigma^{(1)}_{(i,j)}$: $\vec{v} = -\mathbf{M^{-1}\Sigma}\vec{V}$. Now we plug this solution into the first set of equations and get $(\mathbf{D} - \mathbf{\Sigma^T M^{-1} \Sigma})\vec{V} = \vec{I}$. Hence, by comparison with equation (2), we find that

$$\mathbf{L^{(1)}} = \mathbf{D} - \mathbf{\Sigma^T M^{-1} \Sigma},$$

where the left-hand side is understood to depend on $\sigma^{(2)}_{(i,j)}$, and the right-hand side on $\sigma^{(1)}_{(i,j)}$. The decimation map $\mathcal{D}$ is given by the off-diagonal entries:

$$\sigma^{(2)}_{(i,j)} = -(\mathbf{L^{(1)}})_{ij} = (\mathbf{\Sigma^T M^{-1} \Sigma})_{ij}. \tag{7}$$

In general, $(M^{-1})_{\alpha\beta} = P_{\alpha\beta}/\Delta$, where $P_{\alpha\beta}$ is a homogeneous polynomial of degree $n(n-3)/2$ in the variables $\{\sigma_{(i,j)}\}$, while $\Delta = \det(\mathbf{M})$ is a homogenous polynomial of degree $n(n-1)/2$ in the variables $\{\sigma_{(i,j)}\}$.

From these considerations it follows that $\mathcal{D}$ is an $n(n-1)/2$-dimensional rational map from $\mathbb{C}_\infty^{n(n-1)/2}$ to $\mathbb{C}_\infty^{n(n-1)/2}$ in the variables $\{\sigma_{(i,j)}\}$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$; we refer the reader to the specific literature [11] for the language of dynamical systems and rational maps. By construction, $\mathcal{D}$ is homogeneous of degree 1 in its variables: $\mathcal{D}(\lambda\vec{\sigma}) = \lambda\mathcal{D}(\vec{\sigma})$, $\lambda \in \mathbb{C}$. This property allows us to recover a physical meaning for those points with a negative real value (the physical constraint on the admittance σ of a passive element being that $\mathrm{Re}(\sigma) \geqslant 0$). Indeed, if a result holds for a $\vec{\sigma}$, it also holds for all $\lambda\vec{\sigma}, \lambda \in \mathbb{C}$. Hence, a vector $\vec{\sigma}$ is 'physically meaningful' provided that there exists a $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda\sigma_\alpha) \geqslant 0\,\forall\alpha$. For instance, a vector such as $\vec{\sigma} = (-1, 1, 1, \ldots, 1)$ makes sense since it can be multiplied, e.g., for i to give $(-i, i, i, \ldots, i)$ (a set of capacitive and inductive admittances), while the vector $(1+i, 1-i, -1, \ldots)$ cannot be mapped by multiplication into any physically meaningful point. The requirement a set of admittances must satisfy to have a physical meaning is that the set of the vectors representing them in the complex plane covers an angle $\leqslant 180°$.

The number of terms in expression (7) grows very fast with n (approximately as $n!$): for example, the number of terms in $\Delta = \det(\mathbf{M})$ is 7 for $n = 3$, 293 for $n = 4$, 61763 for $n = 5$. Hence, the map is impossible to study as it is. We will focus here on its asymptotic properties. First, we can show that the *isotropic point at* 0 is an attracting fixed point for $\mathcal{D}$. Second, by perturbation of the isotropic solution, we can find the exact asymptotic expansion of $\mathcal{D}$ near the fixed point for any n.

3.2. The isotropic fractal and fixed points

In the isotropic fractal all the admittances have the same value:

$$\sigma_{(i,j)} = \sigma_0 \qquad \forall i, j.$$

Matrix $\mathbf{M} \equiv \mathbf{M_0}$ gets the general form

$$\begin{cases} (M_0)_{\alpha\alpha} = 2(n-1)\sigma_0 & \text{for diagonal entries} \\ (M_0)_{\alpha\beta} = -\sigma_0 & \alpha \text{ and } \beta \text{ adjacent} \\ (M_0)_{\alpha\beta} = 0 & \alpha \text{ and } \beta \text{ not adjacent.} \end{cases}$$

The general form of its inverse $\mathbf{M_0^{-1}}$ is

$$\begin{cases} \left(M_0{}^{-1}\right)_{\alpha\alpha} = \dfrac{n+6}{2n(n+2)}\sigma_0^{-1} & \text{diagonal entries} \\ \left(M_0{}^{-1}\right)_{\alpha\beta} = \dfrac{3}{2n(n+2)}\sigma_0^{-1} & \alpha \text{ and } \beta \text{ adjacent} \\ \left(M_0{}^{-1}\right)_{\alpha\beta} = \dfrac{2}{2n(n+2)}\sigma_0^{-1} & \alpha \text{ and } \beta \text{ not adjacent,} \end{cases}$$

as can be checked by direct multiplication. The general form of $-\mathbf{L_0^{(1)}} = \mathbf{\Sigma_0^T M_0^{-1} \Sigma_0}$ is

$$\begin{cases} \left(-\mathbf{L_0^{(1)}}\right)_{ii} = \dfrac{2n-2}{n+2}\sigma_0 & \text{diagonal entries} \\ \left(-\mathbf{L_0^{(1)}}\right)_{ij} = \dfrac{n}{n+2}\sigma_0 & \text{off-diagonal entries.} \end{cases}$$

By looking at the off-diagonal entries of $-\mathbf{L_0^{(1)}}$, we find that $\mathcal{D}$ maps isotropic vectors into isotropic vectors with a scaling factor of $n/(n+2)$:

$$\mathcal{D}(\vec{\sigma}_0) = \frac{n}{n+2}\vec{\sigma}_0,$$

where $\vec{\sigma}_0 = (\sigma_0, \sigma_0, \ldots, \sigma_0)$. The isotropic point at 0 is an attractor for map $\mathcal{D}$, with the leading term of the asymptotic behaviour:

$$\sigma_\alpha^{(g)} \sim \left(\frac{n}{n+2}\right)^g \sigma_0 \qquad \forall\,\alpha \quad \text{for } g \to \infty, \tag{8}$$

where g is the generation of the fractal and σ_0 depends on the initial conditions. Recalling that the linear size of the fractal scales as 2^g, the exponent $\bar{\zeta} = \ln[(n+2)/n]/\ln 2$ for the scaling of the isotropic conductance introduced in section 1 is recovered.

3.3. Perturbation of the isotropic fractal and asymptotic expansion

We now add a small perturbation to the isotropic fractal by incrementing variable $\sigma_{(1,2)}$ by a value ε:

$$\sigma_{(1,2)} = \sigma_0 + \varepsilon, \qquad \text{while} \quad \sigma_{(i,j)} = \sigma_0 \quad \text{for} \quad (i,j) \neq (1,2),$$

with ε/σ_0 small. $\mathbf{M}$ and $\boldsymbol{\Sigma}$ are increased by infinitesimal matrices: $\mathbf{M} = \mathbf{M_0} + \boldsymbol{\delta}\mathbf{M}$; $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 + \delta\Sigma$, and the equation for $-\mathbf{L^{(1)}}$ becomes

$$\begin{aligned} -\mathbf{L^{(1)}} &= \boldsymbol{\Sigma}^{\mathbf{T}}\mathbf{M}^{-\mathbf{1}}\boldsymbol{\Sigma} \sim (\boldsymbol{\Sigma}_\mathbf{0} + \delta\Sigma)^T\,(\mathbf{M_0} + \boldsymbol{\delta}\mathbf{M})^{-1}\,(\boldsymbol{\Sigma}_\mathbf{0} + \delta\Sigma) \\ &\sim \left(\boldsymbol{\Sigma}_\mathbf{0}^\mathbf{T} + \delta\Sigma^\mathbf{T}\right)\left(\mathbf{I} - \mathbf{M_0^{-1}}\boldsymbol{\delta}\mathbf{M}\right)\mathbf{M_0^{-1}}\,(\boldsymbol{\Sigma}_\mathbf{0} + \delta\Sigma) \\ &\sim \underbrace{\boldsymbol{\Sigma}_\mathbf{0}^\mathbf{T}\mathbf{M_0^{-1}}\boldsymbol{\Sigma}_\mathbf{0}}_{\text{isotropic solution}} + \underbrace{\boldsymbol{\Sigma}_\mathbf{0}^\mathbf{T}\mathbf{M_0^{-1}}\delta\Sigma + \delta\Sigma^\mathbf{T}\mathbf{M_0^{-1}}\boldsymbol{\Sigma}_\mathbf{0} - \boldsymbol{\Sigma}_\mathbf{0}^\mathbf{T}\mathbf{M_0^{-1}}\boldsymbol{\delta}\mathbf{M}\mathbf{M_0^{-1}}\boldsymbol{\Sigma}_\mathbf{0}}_{\text{perturbation}} \\ &= -\mathbf{L_0^{(1)}} - \boldsymbol{\delta}\mathbf{L^{(1)}}. \end{aligned} \tag{9}$$

Knowing the explicit expressions of $\boldsymbol{\delta}\mathbf{M}$ and $\delta\Sigma$, the calculation of the terms contributing to $-\boldsymbol{\delta}\mathbf{L^{(1)}}$ is lengthy but not difficult. We omit here the details and give only the final result. $-\boldsymbol{\delta}\mathbf{L^{(1)}}$ is a symmetric $n \times n$ matrix with entries:

$$\begin{cases} -\dfrac{3n+2}{(n+2)^2}\varepsilon & \text{diagonal entries } (1,1) \text{ and } (2,2) \\ \dfrac{5n-2}{(n+2)^2}\varepsilon & \text{entries } (1,2) \text{ and } (2,1) \\ \dfrac{n-2}{(n+2)^2}\varepsilon & \text{entries } (1,i) \text{ and } (2,i),\, i \neq 1,2 \\ \dfrac{-2}{(n+2)^2}\varepsilon & \text{entries } (i,j), \text{ with } i,j \neq 1,2. \end{cases}$$

By looking at the off-diagonal entries, we see that by incrementing $\sigma_{(1,2)}$ the admittances undergo the following changes:

$$\begin{cases} \sigma_{(1,2)}: & \sigma_0 + \varepsilon \longrightarrow \dfrac{n}{n+2}\sigma_0 + \dfrac{5n-2}{(n+2)^2}\varepsilon \\ \sigma_{(1,i)}, \sigma_{(2,i)}: & \sigma_0 \longrightarrow \dfrac{n}{n+2}\sigma_0 + \dfrac{n-2}{(n+2)^2}\varepsilon \\ \sigma_{(i,j)}: & \sigma_0 \longrightarrow \dfrac{n}{n+2}\sigma_0 - \dfrac{2}{(n+2)^2}\varepsilon. \end{cases}$$

This means that in general, by incrementing *each* admittance σ_α by its own infinitesimal quantity near the isotropic point $\sigma_\alpha \to \sigma_0 + \varepsilon_\alpha$ (in vector notation $\vec{\sigma} \to \vec{\sigma}_0 + \vec{\varepsilon}$), and labelling with α' the links adjacent to α, with α'' the links not adjacent to α, the map changes the increments in the following way:

$$\varepsilon_\alpha \longrightarrow \underbrace{\frac{5n-2}{(n+2)^2}\varepsilon_\alpha}_{\text{diagonal term}} + \sum_{\alpha'} \underbrace{\frac{n-2}{(n+2)^2}\varepsilon_{\alpha'}}_{\text{adjacent links}} + \sum_{\alpha''} \underbrace{\frac{-2}{(n+2)^2}\varepsilon_{\alpha''}}_{\text{non-adjacent links}} .$$

In conclusion, in the vicinity of the isotropic point $\mathcal{D}$ gets the linearized form:

$$\mathcal{D}(\vec{\sigma}_0 + \vec{\varepsilon}) \sim \frac{n}{n+2}\vec{\sigma}_0 + \mathbf{E}\vec{\varepsilon}, \tag{10}$$

where $\mathbf{E}$ is a perturbation matrix with entries:

$$\begin{cases} E_{\alpha\alpha} = \dfrac{5n-2}{(n+2)^2} & \text{diagonal entries} \\ E_{\alpha\beta} = \dfrac{n-2}{(n+2)^2} & \alpha \text{ and } \beta \text{ adjacent} \\ E_{\alpha\beta} = \dfrac{-2}{(n+2)^2} & \alpha \text{ and } \beta \text{ not adjacent.} \end{cases} \tag{11}$$

The following section is devoted to the study of the eigenvalues and eigenspaces of $\mathbf{E}$.

4. Eigenvalues and eigenspaces of the linearized map

The basic result is that matrix $\mathbf{E}$ has three eigenvalues e_1, e_2, e_3 for any n (with e_3 disappearing only for $n = 3$). Their values and multiplicities are:

eigenvalue	multiplicity
$e_1 = \dfrac{n}{n+2}$	1
$e_2 = \dfrac{n(n+1)}{(n+2)^2}$	$n-1$
$e_3 = \dfrac{3n}{(n+2)^2}$	$\dfrac{1}{2}n(n-3).$

We call $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ the subspaces spanned by e_1, e_2, e_3, respectively.

The isotropic vector with 1 on all the links is an eigenvector of $\mathbf{E}$ (since all rows are permutations of each other). The eigenvalue is the sum of the entries of a row, which is easily shown to be $\frac{n}{n+2}$. This could be expected from equation (3.2), since an equal increment of ε for all admittances just shifts the system to the different isotropic point $\sigma_0 + \varepsilon$.

4.1. *Eigenvectors corresponding to e_2: Wheatstone's conditions*

The eigenvectors for eigenvalue e_2 can be built as follows. Take two vertices of the graph, for instance 1 and 2, and consider the link $(1, 2)$ joining them. Then build a vector this way: all the $n-2$ links incident with 1, *except* link $(1, 2)$, are given the value 1; all the $n-2$ links incident with node 2, except link $(1, 2)$, are given the value -1; link $(1, 2)$ and all the other links have value 0. We call $\vec{\mathsf{f}}_{(1,2)}$ this vector; in general we call $\vec{\mathsf{f}}_{(i,j)}$ the vector built in this way starting from link (i, j). Figure 6 shows two of these eigenvectors for the graph K_5.

$\vec{\mathsf{f}}_{(i,j)}$ can be shown to be an eigenvector for matrix $\mathbf{E}$ by direct multiplication for the matrix. An easier and physically more interesting way is seeing that for vectors $\vec{\sigma}_0 + \varepsilon\vec{\mathsf{f}}_{(i,j)}$

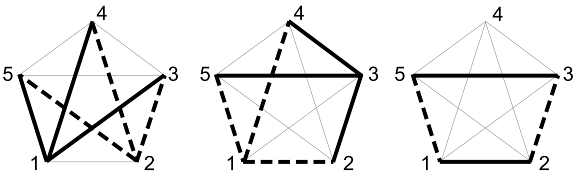

Figure 6. Some eigenvectors of the matrix **E** of the linearized map for the K_5 graph: thin lines denote links with value 0, thick continuous lines denote links with value 1 and thick dotted lines denote links with value -1. Vectors $\vec{f}_{(1,2)}$ and $\vec{f}_{(1,3)}$, which are eigenvectors for the eigenvalue e_2, are shown on the first two graphs from the left. A 4-loop with alternating 1 and -1 on its links is shown on the graph on the right: it is an eigenvector for eigenvalue e_3.

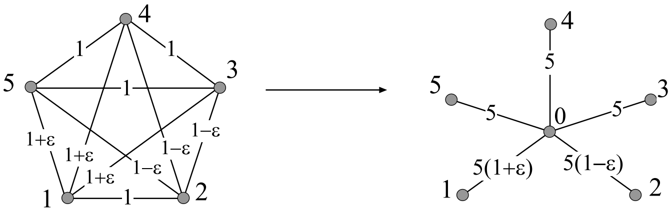

Figure 7. A K_5 graph with $\sigma_0 = 1$ perturbed with an eigenvector $\varepsilon\vec{f}_{(1,2)}$ corresponding to eigenvalue e_2 (left). The circuit satisfies Wheatstone's conditions to first order in ε: in every subgraph forming a quadrilateral the products of opposite admittances are equal. A generalized mesh-star transformation is thus possible (right).

Wheatstone's conditions mentioned in section 2 hold to first order; hence, a generalized mesh-star transformation is valid (figure 7). The star equivalent to this mesh has an admittance equal to $n(\sigma_0 + \varepsilon)$ for links $(0, i)$ (0 being the additional central node), $n(\sigma_0 - \varepsilon)$ for links $(0, j)$ and $n\sigma_0$ on all the other links. The decimation can then be carried out with the same easy topological procedures described in [9] for the 3-simplex lattice, yielding $\frac{n}{n+2}\vec{\sigma}_0 + \frac{n(n+1)}{(n+2)^2}\varepsilon\vec{f}_{(i,j)}$.

It can be easily shown that the set $\vec{f}_{(1,j)}$ (with $i = 1$ fixed and $j = 2, \ldots, n$) is a set of linearly independent (not orthogonal) vectors. Furthermore, any other $\vec{f}_{(k,l)}$ can be obtained from this set by means of the formula $\vec{f}_{(k,l)} = \vec{f}_{(1,l)} - \vec{f}_{(1,k)}$. Hence, the number of independent vectors of this eigenspace (the multiplicity of the eigenvalue e_2) is $n - 1$.

The eigenvalue e_2 corresponds to the well-known anisotropy exponent $\bar{\lambda} = \ln(e_1/e_2)/\ln 2 = \ln[(n+2)/(n+1)]/\ln 2$ discussed in the introduction. However, the dimension of the eigenspace $\mathcal{E}_2$ and the Wheatstone property were not known up to now.

4.2. Eigenvectors corresponding to e_3: even-cycle space and crossover property

In the graph theory, a *cycle* of length ℓ, or ℓ-cycle, is a closed walk (sequence of connected vertices) composed of ℓ distinct points and links. The eigenvectors for eigenvalue e_3 are *all the cycles of even length with alternating* 1 *and* -1 *on their links*, and 0 on the links not belonging to the cycle (we will call them simply 'even cycles' henceforth, with the implicit assumption on the value of their links). One such eigenvector for K_5 is shown in figure 6, on the right, in the case of a four-cycle. In this case, the property of being an eigenvector for **E** has to be verified by direct multiplication.

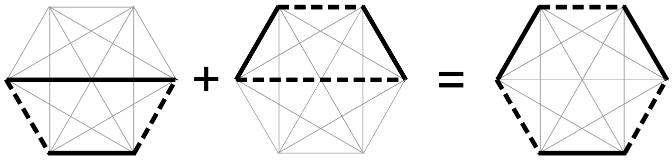

Figure 8. A six-cycle obtained from two four-cycles in the way shown above in a K_6 graph. The same is true for any even cycle: four-cycles form a good basis for the space $\mathcal{E}_3$.

The eigenvalue e_3 corresponds to a *secondary* anisotropy exponent with value $\bar{\mu} = \ln(e_1/e_3)/\ln 2 = \ln[(n+2)/3]/\ln 2$. The new exponent appears when the contribution of the first anisotropy exponent $\bar{\lambda}$ is negligible, that is, if we start from a configuration orthogonal (or almost orthogonal) to the space $\mathcal{E}_2$.

Since a generic even cycle can be obtained from a linear combination of four-cycles (figure 8), four-cycles are a basis for the space $\mathcal{E}_3$. The number $\mathcal{N}_{4,n}$ of linearly independent four-cycles on a K_n graph (the multiplicity of e_3) can be computed (for example, by recurrence starting from a K_4 graph) and is found to be $\mathcal{N}_{4,n} = \frac{1}{2}n(n-3)$. For $n = 3$, when the basic cell is a triangle (and the fractal is the usual two-dimensional Sierpiński gasket), there are no even loops and the only eigenvalues are e_1 and e_2 [9]. For $n \geqslant 5$, the eigenspace $\mathcal{E}_3$ is that with the largest dimensionality, growing as n^2 for $n \gg 1$, as opposed to that of $\mathcal{E}_2$ that grows as n.

When the starting configuration is not exactly orthogonal to $\mathcal{E}_2$, the behaviour of the projection of $\vec{\sigma}$ onto $\mathcal{E}_3$ displays an interesting crossover property, due to the homogeneity of the map $\mathcal{D}$, which makes the value of $\bar{\mu}$ deviate from that calculated in the first-order approximation. In order to see it, we need to calculate the second-order term in the series expansion for the projection of $\vec{\sigma}$ onto $\mathcal{E}_3$: it typically contains quadratic contributions of the form

$$(\ldots)\frac{\sigma_{(1,2)}^2}{\sigma_0} + (\ldots)\frac{\sigma_{(1,2)}\sigma_{(1,3)}}{\sigma_0} + \cdots,$$

where the $(\ldots)$ denote some constant. Due to σ_0 in the denominator, for $g \to \infty$, the leading term of this expansion is proportional to

$$\left(\frac{e_2^2}{e_1}\right)^g = \left(\frac{n(n+1)^2}{(n+2)^3}\right)^g > (e_3)^g.$$

Hence, the asymptotic behaviour $\sim(e_3)^g$ normally holds only for some iterations (the number of which depend on the initial conditions), then the second-order term $\sim\left(e_2^2/e_1\right)^g$ prevails. This means that the second anisotropy exponent $\bar{\mu}$ crosses over from $\bar{\mu} = \ln[(n+2)/3]/\ln 2$ to $\bar{\mu} = \ln[(n+2)^3/n(n+1)^2]/\ln 2$.

5. Conclusions

We have extended the method used in [6] to find the decimation map $\mathcal{D}$ for a network of admittances on a generic n-simplex lattice. By means of a linear expansion near the isotropic fixed point of the map, we have found the first-order asymptotic behaviour of $\mathcal{D}$ for every n. The eigenspaces of the linearized map have been found to be always three (two just for $n = 3$), and to have a direct interpretation in terms of graph theory. In particular, the third eigenspace is connected to the set of even-length cycles on the basic cell, and its eigenvalue is related to a secondary anisotropy exponent $\bar{\mu}$ with a value that crosses over from $\ln[(n+2)/3]/\ln 2$ to $\ln[(n+2)^3/n(n+1)^2]/\ln 2$ with the size of the fractal.

Due to a well-established correspondence between electrical networks and random walks [12], our results could be easily extended to the random walk problem on these lattices, with jumping probabilities depending on the direction. Since our method of direct manipulation of the Laplacian matrix is quite general, we are currently studying its application to other exactly-decimable fractals, to non-exactly-decimable fractals (with some proper approximations) and to more general graphs.

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